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Geometry of a Visual Illusion*

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A new visual illusion is predicted from the assumption that the perceived distance along any path depends on the discriminability for position along the path. A disk is placed between two dots, so that the straight path between the dots is nearly tangent to the disk. It is predicted that, for the perceived straight path between the dots to be tangent to the disk, the disk must overlap the physically straight path between the dots by an amount proportional to its radius. Furthermore, certain patternings of the disk are predicted to reduce the amount of illusion. All predictions are confirmed in detail. The results are compared with those obtained in an experiment on the filled-space illusion.

INTRODUCTION

THE possibility that discriminability is an appropriate measure of perceived distance or magnitude has been both suggested[1,2] and denied.[3] The use of discriminability as a measure of perceived distance is not equivalent to a restatement of Fechner's law, because while two just noticeable differences have equal discriminability, it is not necessarily true that two intervals of equal discriminability contain the same number of just noticeable differences. Discriminability is taken here as a generalization of Tanner's d' measure of signal detectability.[4] It is also nearly equivalent to Garner's idea of discriminability, as used in his scales of equal discriminability.[5] The difference between discriminability as used here and discriminability as used by Garner is due to the probable anchoring effect of the stimulus in an "absolute judgement" type of experiment. Such anchoring effects for visual position are discussed by Taylor.[6]

On the assumption that discriminability is directly related to perceived distance, several predictions have been made,[2] for one of which the present experiment provides a test.

PREDICTIONS

Suppose that two dots are placed in an otherwise empty space. The perceived distance between them will have some finite value, depending on the discriminability for position along the shortest path connecting them. If, now, a mark is made on that shortest path, the discriminability for position will increase in the neighborhood of the mark, and the path will seem longer. If a second mark is placed on the path, it will again increase in length, provided that the position of one mark is appropriate relative to the position of the other. In particular, consider the case where the two marks are symmetrically placed relative to the midway point in the path. If the two marks are coincident, their joint effect will be that of one mark alone. If they are so placed as to trisect the distance, their joint effect will be greater than that of either alone.

Just as these predictions apply to the straight path between two dots, so they apply also to any curvilinear path between the same two dots. Anchoring points on the path increase the perceived length of the path, by assumption. Improved discriminability along a path may result from the presence of markings near the path, as well as from those on it. Consider the case where a homogeneous circular disk is placed near, but not across, the straight path connecting two dots, with the two dots equidistant from the center of the disk. Discriminability for position along the path will be aided by the presence of the disk. If the disk is very large, so that its edge runs nearly parallel to the path for a considerable distance, it will aid discriminability very little; whereas if it is small, and the same distance from the path, it will be of more assistance to discriminability. The opposite relation holds if the disk actually crosses the path. In this case there are two anchoring points where the path cuts the disk edge and, within limits, the further apart these anchoring points are, the more discriminability is increased. Hence, for a given amount of overlap large circles result in higher discriminability than small circles.

It is possible to quantify these relationships. The discriminability for position along a given path depends on where the path passes relative to the disk. If the path deviates only slightly from straightness, its position relative to the disk is specified entirely by any two of R, B, and Y [see Fig. 1(a)]. It will be found convenient to use B and Y, since R does not vary among paths about any given disk.

The perceived length of the path will depend also on its physical length M, where $M=L+\Delta L$ [see Figs. 1(b) and 1(c)], and on its curvature, an index of which is

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1 Anonymous, "Popular Scientific Recreations, a storehouse of instruction and amusement: in which the Marvels of Natural Philosphy, Chemistry, Geology, Astronomy, etc., are explained and illustrated, mainly by means of pleasing experiments and attractive pastimes." Translated and enlarged from Gaston Tissandier, *Les Recreations Scientifiques*, New and Enlarged Edition [Ward, Lock, Bowden and co., London, 1882 (?)], p. 113. The statement does not appear in the original.

2 W. R. Garner, J. Exptl. Psychol. **43**, 232–238 (1952); J. Acoust. Soc. Am. **30**, 1005–1012 (1958); M. M. Taylor, Can. J. Psychol. (to be published).

3 S. S. Stevens and E. H. Galanter, J. Exptl. Psychol. **54**, 377–411 (1957); S. S. Stevens, Am. J. Psychol. **71**, 633–646 (1958).

4 W. P. Tanner, J. Acoust. Soc. Am. **30**, 922–928 (1958); see also D. M. Green, J. Acoust. Soc. Am. **32**, 1189–1202 (1960).

5 W. R. Garner, reference 2 (1952).

6 M. M. Taylor, Perceptual and Motor Skills **12**, 203–30 (1961).

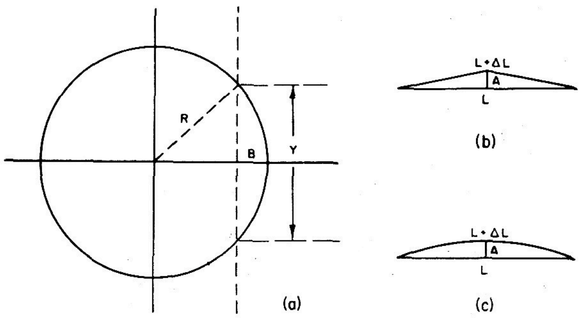

FIG. 1. Constructions for determining the path of the tangent geodesic.

given by A [see Figs. 1(b) and 1(c)]. In all, then, the perceived length of a path is a function of the four variables A, B, Y, and M. It should be emphasized that perceived length is never actually determined in the following argument, but it is assumed to have a functional dependence on the four variables mentioned, including physical path length. It will be an even function of Y and of M, since it cannot depend on their sign, so that it may be written

$$D=D(A,B,Y^2,M^2),$$

where D is the perceived length of the path.

Now, if the perceived straight path is the path perceived as shortest between the given end points, then its derivative with respect to a change in physical bend must vanish. Symbolically,

$$dD/dA=0 \tag{1}$$

for any path perceived as straight, if the perceptual geometry is Riemannian.

In general, for any smooth function, $F(x,y_1,y_2\cdots)$,

$$dF/dx=\partial F/\partial x+(\partial F/\partial y_1)(dy_1/dx) +(\partial F/\partial y_2)(dy_2/dx)+\cdots, \tag{2}$$

so that

$$dD/dA=\partial D/\partial A+(\partial D/\partial B)(dB/dA) +(\partial D/\partial Y^2)(dY^2/dA)+(\partial D/\partial M^2)(dM^2/dA), \tag{3}$$

which expresses the variation in perceived distance with a change in the physical parameter A. Some of the terms in this general expression may be written directly in terms of the physical variables.

If the end points of the path are held constant in position relative to the disk, as they are in the experiment, then as A increases, B decreases by the same amount, so that

$$dB/dA=-1 \tag{4}$$

and

$$dY^2/dA=(dY^2/dB)(dB/dA)=-(dY^2/dB). \tag{5}$$

Now the equation of the circle of Fig. 1(a) is

$$(R-B)^2+(Y/2)^2=R^2,$$

from which

$$Y^2=8RB-4B^2,$$

so that

$$dY^2/dB=8(R-B)$$

and

$$dY^2/dA=-8(R-B). \tag{6}$$

Without knowledge of the exact form of the physical curve followed by the perceived straight path, dM^2/dA cannot be exactly specified. However, a close approximation can be made. It is possible to show that for the curve of Fig. 1(b), for small A,

$$\Delta L \doteq 2A^2/L$$

while for that of Fig. 1(c),

$$\Delta L \doteq 8A^2/3L.$$

In either case,

$$d\Delta L/dA \doteq kA/L. \tag{7}$$

This form of relation can be shown to hold if the curve is reasonably smooth and if A is small. The value of k depends on the exact form of the curve but is usually about 5. Now

$$d\Delta L/dA=d(L+\Delta L)/dA=dM/dA$$

and

$$dM^2/dA=2M(dM/dA)\doteq 2L(dM/dA)$$

from (7)

$$=2kA. \tag{8}$$

All of the expressions (4) to (8) concern relations in physical space. When the indicated substitutions are made in (3),

$$dD/dA=\partial D/\partial A-\partial D/\partial B -8(R-B)\partial D/\partial Y^2+kA(\partial D/\partial M^2) \tag{9}$$

from (1) $=0$. This relation will hold in general for any path perceived as straight, whether or not the path actually cuts the disk. The functional relationship holds whether Y is real or imaginary, whether B is positive or negative. It specifies what relations among the physical variables must hold for the particular path to be perceived as straight.

The expression (9) may be further simplified. If D is an even, smooth function of A, then $\partial D/\partial A=0$, for $A=0$. If A is small, then $\partial D/\partial A$ is also small, and may be assumed equal to zero for purposes of calculation. This assumption may be tested by direct experiment.

If the path perceived as straight is also perceived as tangent to the disk, so that $B=0$, another simplification is possible. It seems likely that the effect of positive B is similar to the effect of negative B, especially when B is small. If this is true, then for $B=0$, $\partial D/\partial B$ may be assumed to vanish.

During any given experimental series, L is held constant, so that $\partial D/\partial M^2$ will be constant, since M is approximately equal to L. Similarly, for a tangent path, Y is always zero, so that $\partial D/\partial Y^2$ is a constant throughout. When all these simplifications are made, the expression (9) reduces to

$$cA = R, \tag{10}$$

where

$$c = 2k(\partial D/\partial M^2)_{M=L}/8(\partial D/\partial Y^2)_{Y^2=0},$$

which is constant. In words, for the perceptually straight path between two given dots to be perceived as tangent to a disk, the disk must overlap the physically straight path between the dots by an amount proportional to its radius, at least for disks of small radius. Since the two partial derivatives of perceived distance are constants under variation of disk radius, the form of the relation between D and the physical variables does not enter the argument.

The proportionality constant c is inversely proportional to $(\partial D/\partial Y^2)_{Y^2=0}$. If the anchoring effect of the disk edge is reduced, $(\partial D/\partial Y^2)_{Y^2=0}$ will also be reduced, raising the factor c, which means that the amount of overlap necessary for the path to be perceived as tangent will also be reduced for a given radius of disk. The anchoring effect of the disk edge can be reduced by diffusing the edge in some way, such as by using a speckled disk of random dots rather than a solid disk. The anchoring effect also will be smaller for a subject (S) who does not make use of all the information in the pattern than for one who can make good use of the available information. Differences in discriminative ability are well known to exist between Ss. Subjects who are poor discriminators should then show poor consistency and low illusion as compared with Ss who are better discriminators.

The amount of illusion can be altered in a different way. If the disk is striped, with the stripes perpendicular to the line between the dots, then projections of the stripes beyond the disk edge provide some anchoring. The assumption that $\partial D/\partial B = 0$ will then be false. The analysis will not apply directly, because of the extra anchoring points, but it is apparent that the rate of change of discriminability with position relative to the circle edge will be lessened. A smaller illusion will result.

There are, then, three predictions about the behavior of this illusion. (1) A solid disk must overlap the straight path by an amount proportional to its radius in order for the visually straight path between the dots to appear tangent to the disk. (2) Less consistent Ss should show less illusion than is shown by more consistent Ss. (3) Both striped and speckled disks should yield less illusion than solid disks.

EXPERIMENTAL TEST OF THE PREDICTIONS

The typical test pattern was as shown in Fig. 2. There were nine sets of test patterns, which differed from each other in the size and patterning of the disk. Test patterns within a set differed from each other only in the amount by which the disk overlapped the physically straight line between the dots marked "A" in Fig. 2. The other features of the test pattern were used for

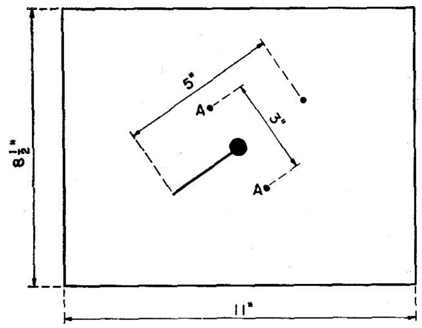

FIG. 2. Typical test pattern used in the study. The line was 0.015 in. thick, and all dots were about 0.01 in. diam. Test patterns were on 8½ by 11-in. "Albanene" tracing paper.

another study, though they were present in this one. There were between 6 and 9 members per set of test patterns. Six sets of test patterns had a solid black disk, cut from a thin plastic material called "Zip-a-Tone." The diameters of these solid disks were 0.1, 0.25, 0.5, 0.75, 1.0, and 1.5 in. One set had no disk on the end of the line. For purposes of discussion, this set was considered to have a solid disk equal in diameter to the width of the line, 0.015 in. One set had disks of speckled Zip-a-Tone, 1.5 in. diam. The speckles covered about $\frac{1}{6}$ of the area of the disk, at about 2400 dots/in^2. The final set had disks of 1.5 in. diam., made with striped Zip-a-Tone laid over the speckled material, stripes perpendicular to the physically straight line between the dots. The stripes were 0.04 in. wide, 0.1 in. apart.

The amount of illusion was obtained by the method of constant stimuli. S was shown the test patterns in irregular order, and told to judge for each whether or not the disk overlapped the straight line between the dots. "Tangent" judgments were not allowed. Each S judged each stimulus once in each of four orientations, and the total numbers of each response, summed over all Ss and all orientations were used in a probit analysis to determine the mean illusion for all Ss combined, for each type of test pattern. A similar analysis was done visually for each S individually. Ss were 14 members of the staff of the Defence Research Medical Laboratories.

The results are shown in Figs. 3 and 4. In Fig. 3 is shown the amount of physical overlap which yields equal numbers of each judgment, for each type of test pattern. The plotted points are obtained from the pooled data from all Ss. One standard error of the mean is indicated with each point. For the solid circle, the amount of overlap is proportional to the diameter of the circle with a constant of proportionality of 0.072. A straight line through the origin accounts for all but a nonsignificant amount of variability. There is no significant difference in the variability about the different points. The illusion is less for the speckled circle than for a solid circle of the same size, and still less for the striped circle. These results agree with predictions.

The amount of illusion for individual Ss is shown in Fig. 4, as a function of their consistency. The amount of

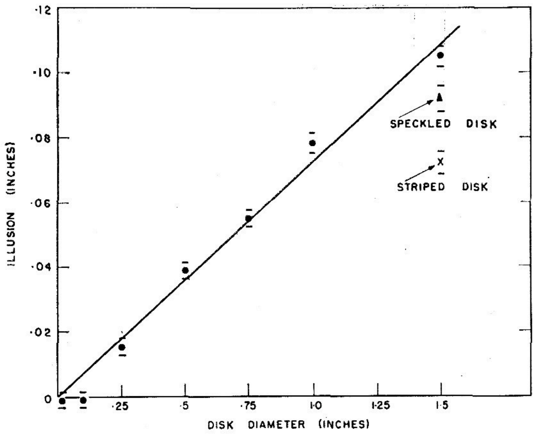

FIG. 3. Relation for solid disks between amount of illusion and disk diameter. Results are also shown for striped and speckled disks.

illusion was obtained by plotting a graph similar to that of Fig. 3 for each S, and drawing in a best-fit line by eye to get the amount of illusion for a disk of 1.0 in. diam. The number of reversals, which is taken as a measure of consistency, records the number of times during the entire experiment with the solid disks that the particular S claimed one disk overlapped, when another of the same type at the same orientation but with more physical overlap was seen as not overlapping. The more reversals, the less consistent the S. It is clear from Fig. 4 that the more consistent Ss showed more illusion, as predicted.

RELATION TO THE FILLED-SPACE ILLUSION

The results of the experiments with the solid disks follow the predicted straight line so well that it is tempting to press the analysis a little further. The equation

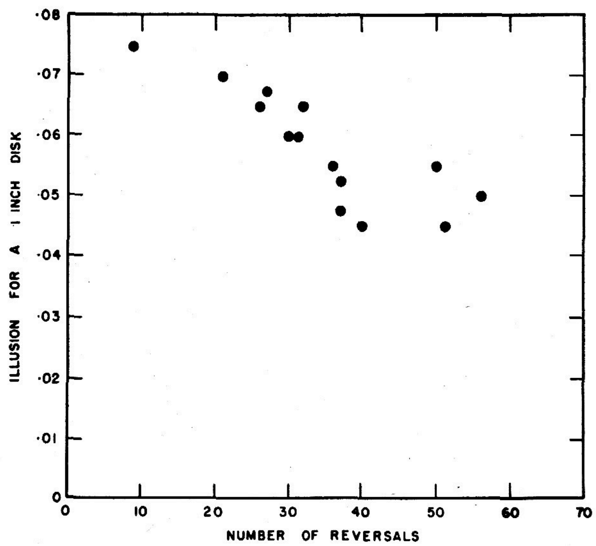

FIG. 4. Relation between consistency and amount of illusion for individual Ss. More reversals indicate less consistency.

for the straight line is[7]

$$A/R=(8/k)(L\partial D/\partial Y^2)$$

or

$$\partial D/\partial Y^2=kA/8RL.$$

From the experimental results, A/R is about 0.14, k about 5 if the true geodesic path is reasonably like an arc of a circle, and L is 3 in., or about 0.15 rad subtended at the eye. From these figures,

$$\partial D/\partial Y^2=0.7 \text{ rad}^{-1} \text{ at } Y=0.$$

It should be possible to test this prediction by direct experiment. Unfortunately, there seem to be no data available bearing directly on the question. The most directly applicable seem to be those of Spiegel,[8] who performed a comprehensive study of the filled-space illusion.

Spiegel presented his Ss with a display consisting of short vertical lines of light arranged so that their midpoints were all at the same level. Four lines were of the same height and width. Two served to define a comparison distance; the other two, a test interval. Shorter and thinner lines were interpolated between the lines marking the test interval. Subject was required to adjust the comparison interval to apparent equality with the test interval. Four different physical test intervals were used, with many different combinations for the interpolated lines.

Unfortunately, for the comparison with the present study, Spiegel did not include a condition in which there were two closely spaced interpolated lines (equivalent to Y small, in the present study) near the midpoint of the test interval. The nearest condition used by Spiegel subtended 0.14 rad at the eye between end points. Only two relevant data points are available, for $Y=0$, and by interpolation for $Y=L/3$. These two points are not really satisfactory, since D may be expected to vary little with Y in the neighborhood of $Y=L/3$. The obtained value of $\partial D/\partial Y^2$ is expected to be somewhat lower than the value of $\partial D/\partial Y^2$ for $Y=0$. The difference in matched length between these two conditions was found by Spiegel to be about 0.0044 rad. Taking $Y=0.047$ rad, $\partial D/\partial Y^2$ is about 2.0 rad^{-1}, which is somewhat higher than predicted from the results of the present study. The true value of for $\partial D/\partial Y^2$ at $Y=0$ will be higher yet.

It is not very surprising that the value for $\partial D/Y^2\partial$ should be higher in a direct experiment on the filled-

[7] This expansion may be obtained from the basic equation by using M instead of M^2, so that the term $(\partial D/\partial M^2)(dM^2/dA)$ is replaced by

$$(\partial D/\partial M)(dM/dA)=kA/L(\partial D/\partial M).$$

$\partial D/\partial M$ is always evaluated for $M\doteq L$, so that it is a constant, and this constant will probably not be far from unity, compared with the range of uncertainty in the value of k. For purposes of calculation in this section, it is assumed that $\partial D/\partial M=1$.

[8] H. G. Spiegel, Psychol. Forsch. 21, 327–83 (1937). A summary in English of his main result is given by Taylor, reference 6.

space illusion than in the present study. The points of intersection of the path with the circle are not as well defined as are the filling marks in the direct experiment, The aid to discriminability afforded by the filling marks is liable to be higher than that given by the intersections, so that $\partial D/\partial Y^2$ would also be higher. Despite this effect, there is agreement in the order of magnitude between the two values.